QFT

characterise "excitations" of the fields by the way they transform under group action $\begin{pmatrix}\text{translation}\\ \text{rotation}\end{pmatrix}$

spin: 0 scalar	$\frac{1}{2}$ fermion	1 boson

irreducible representations of the Lorentz group*

$$R = \begin{pmatrix} 1 & & \\ & D_2 & \\ & & D_3 \end{pmatrix}$$

$\Downarrow$ if the laws are invariant, then these numbers are constants

mass: $m^2 = E^2 - p^2 = P_\mu P^\mu$, invariant under boosts.

electric charge: particles
anti-particles

$$U(1) \cong SO(2)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = g_\alpha(\vec{\phi})$$

$$\phi = \phi_1 + i\phi_2$$

$$\phi' = \cos\alpha\,\phi_1 - \sin\alpha\,\phi_2 + i(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2)$$
$$= (\cos\alpha + i\sin\alpha)\phi_1 + i(\cos\alpha + i\sin\alpha)\phi_2$$
$$= e^{i\alpha}\phi = g_\alpha(\phi)$$

Had I chosen $\bar{\phi} = \phi_1 + i(-\phi_2) = \phi_1 - i\phi_2$

$$\bar{\phi}' = e^{-i\alpha}\bar{\phi} = g_\alpha(\bar{\phi})$$

$$\frac{\partial g_\alpha}{\partial \alpha}\Big|_{\alpha=0} \phi = \underline{\underline{i}}\,\phi \longrightarrow +$$ this is the particle's "charge"

$$\frac{\partial g_\alpha}{\partial \alpha}\Big|_{\alpha=0} \bar{\phi} = -\underline{\underline{i}}\,\bar{\phi} \longrightarrow -$$ this is the anti-particle's "charge"

(more info when Noether thm)

1) Variational formulation of QFT / QM
   ↳ path integral

2) Transformations from group action
   → continuous ⟺ Lie group ⟺ Lie algebra
   ↳ spacetime transformations (diffeomorphis)
   ↳ internal transformations (flavour)

3) Symmetry ⟺ conserved "charges"
   (Noether theorem)
   ↳ Classify states/solutions by their charges
   ↳ "Recipe" for building models based on symmetry

Classical mechanics (particles)

Quantum mechanics

Lagrangian

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}(q, \dot{q})\, dt$$

?

$$\delta S = 0 \Leftrightarrow \text{E-L}$$

$\longleftrightarrow$

Path integral

↳ semiclassical limit

↳ QFT

$\updownarrow$

$\updownarrow$?

Hamiltonian

(q, p)

$\hat{q}, \hat{p}$ operators

$$H(q,p) \xleftrightarrow{Q1} \hat{H}(q,p)$$

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial p}$$

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\{q, p\} = 1$$

$$[\hat{q}, \hat{p}] = i\hbar$$

$$\frac{df}{dt} = \{f, H\}$$

Heisenberg

Schroedinger

$$\hat{A}(t) = e^{\frac{i\hat{H}t}{\hbar}}\hat{A}(0)e^{-\frac{i\hat{H}t}{\hbar}} \qquad |\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$$

$$\Downarrow$$

$$\dot{\hat{A}} = \frac{i}{\hbar}[\hat{A}, \hat{H}]$$

$$\Downarrow$$

$$\text{Classical fields} \xleftrightarrow[Q2]{} \text{QFT}$$

Path integral formulation of QM

$\Psi(x,t)$ satisfies the Schroedinger equation / Hamiltonian perspective

$$\left(i\hbar \partial_t - H_x \right) \Psi = 0$$

for a free particle $H_x = -\frac{\hbar^2}{2m} \partial_x^2$

with initial condition $\Psi(x,0) = g(x)$

Fundamental solution (owing to heat equation)

$$\left(i\hbar \partial_t - H_x \right) G = 0 \quad , \left[\partial_t - \frac{i\hbar}{2m} \partial_x^2 \right] G = 0$$

$$G(x,0) = \delta(x)$$

$$\Rightarrow \quad G(x,t) = \Theta(t)\, K(x,t)$$

$$K(x,t) = \frac{1}{\sqrt{4\pi \left(\frac{i\hbar}{2m}\right) t}}\, e^{-\frac{x^2}{4 \frac{i\hbar}{2m} t}}$$

free particle

$$= \sqrt{\frac{m}{2\pi i \hbar t}}\, e^{\frac{im}{2\hbar t} x^2}$$

$$\Psi(x,t) = \int \underbrace{K(x-y,t)}_{\text{propagator}}\, \Psi(y,0)\, dy$$

the propagator conveys the probability amplitude

$$K(x_f, t_f, x_i, t_i) = \langle x_f, t_f | x_i, t_i \rangle$$

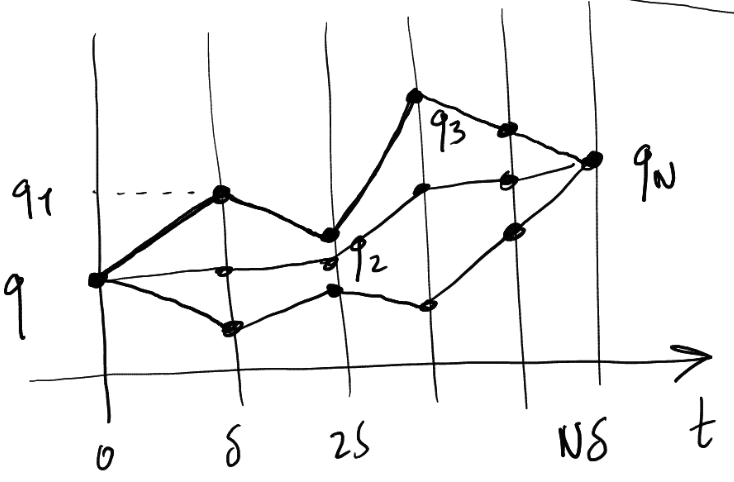
1) $$P_{x_i \to x_f} = \sum_{x_\alpha} P_{x_i \to x_\alpha} P_{x_\alpha \to x_f}$$

$$K(x_f, t_f, x_i, t_i) = \int dy \; K(x_f, t_f, y, T)\, K(y, T, x_i, t_i)$$

Splitting in N portions of $\frac{t_f - t_i}{N} = \delta$

$$\phi = K(q_N, N\delta, q_0, 0) = \underbrace{\int dq_1 \cdots dq_{N-1}}_{\sum_{\gamma \text{ path}}} \; \underbrace{K_{q_N q_{N-1}} K_{q_{N-1} q_{N-2}} \cdots K_{q_1 q_0}}_{\Phi[\gamma]}$$

$$= \sum_{\gamma \text{ path}} \Phi[\gamma]$$

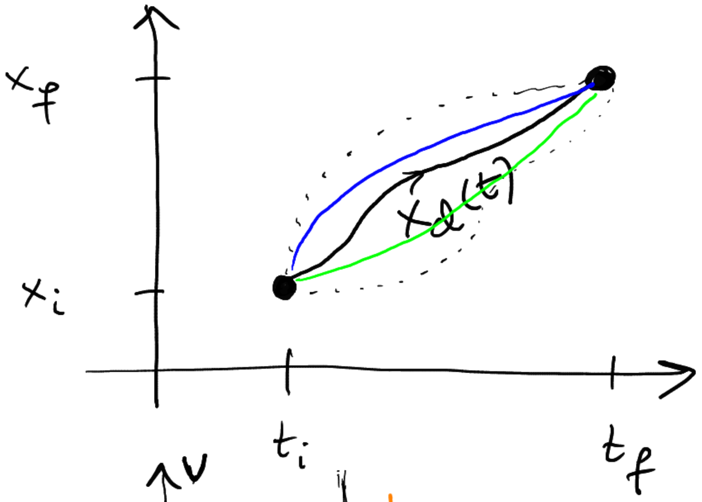


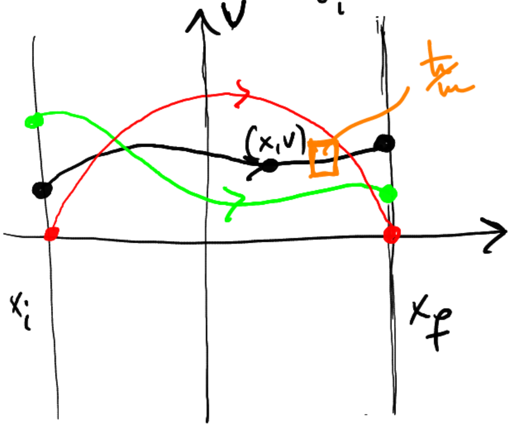
where

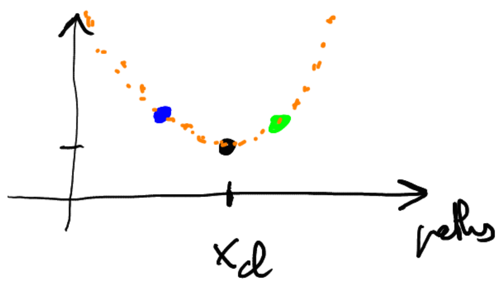
$$\sum_{\text{path}} = \int dq_1 \cdots dq_{N-1}$$

$$\Phi[\gamma] = K_{q_N q_{N-1}} \cdots K_{q_1 q_0} = \phi[\gamma_{N\,N-1}] \cdots \phi[\gamma_1] \quad , \text{ multiplicative}$$

Principal of Least action







The classical path is the one that minimizes the action

$$S[\gamma]_{x_f,x_i} = \int_{t_i}^{t_f} \mathcal{L}(\gamma,\dot{\gamma})\,dt$$

$$\delta S[\gamma_{cl}] = 0$$

In QM, the uncertainty principle states

$$\Delta x \Delta v \lesssim \frac{\hbar}{m}$$

leads to a stochastic (random walk) description of a quantum trajectory

$\Longrightarrow$ probabilities / propagator

2) For a given path (choice of $\{q_k\}$)

$$S[\gamma] = \int_0^{\delta N} \mathcal{L}(q,\dot{q})\,dt = \sum_{k=1}^{N} \int_{(k-1)\delta}^{k\delta} \mathcal{L}\,dt$$

$$= \sum_{k=1}^{N} S[\gamma_{k-1,k}] \quad , \quad \text{additive}$$

$$\Longrightarrow \quad K_{q_{k-1} q_k} \propto e^{\frac{i}{\hbar} S[\gamma_{k-1,k}]}$$

$$K(x_f, t_f, x_i, t_i) = \int \mathcal{D}q(t)\, e^{\frac{i S[q(t)]}{\hbar}}$$

where $S[q] = \int_{t_i}^{t_f} \mathcal{L}(q, \dot{q})\, dt$ and $q(t_i) = x_i$, $q(t_f) = x_f$

---

then, we can evolve the solution by

$$\Psi(x,t) = \int \mathcal{D}[x]\, \Psi(x(0), 0)\, e^{\frac{i S[x]}{\hbar}}$$

Free particle

$$S[\gamma] = \int_0^T \frac{1}{2} m \dot{\gamma}^2 \, dt$$

Let's split $T = N\delta$,

mean value thm (assumption of continuity of $\dot{\gamma}$) mean value thm

$$S_k = \int_{k\delta}^{\delta(k+1)} \frac{1}{2} m \dot{\gamma}^2 \, dt = \frac{\delta m}{2} \dot{\gamma}^2(c) \cong \cancel{\delta} \frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\delta^{\cancel{2}}}$$

$$K(x_f, T, x_i, 0) = A(T) \int dx_1 \cdots dx_{N-1} \, e^{\sum_{k=0}^{N-1} \frac{i}{\hbar} \frac{m}{\delta} \frac{(x_{k+1} - x_k)^2}{2}}$$

Hint:

$$\frac{1}{2}\left[\left(x_0^2 - 2x_0x_1 + x_1^2 \right) + \left(x_1^2 - 2x_1x_2 + x_2^2 \right) + \ldots \right]$$

$$= x_1^2 - 2x_1 \left(\frac{x_0 + x_2}{2} \right) + \frac{x_0^2 + x_2^2}{2} + \sum_{k=2}^{N-1} \frac{(x_{k+1} - x_k)^2}{2}$$

$$= \left(x_1 - \frac{x_0 + x_2}{2} \right)^2 \underline{- \frac{(x_0 + x_2)^2}{4} + \frac{x_0^2 + x_2^2}{2}} + \cdots$$

$$\underline{\qquad} = \frac{1}{4} \left[\cancel{2} x_0^2 + \cancel{2} x_2^2 - \cancel{x_0^2} - \cancel{x_2^2} - 2x_0x_2 \right]$$

$$= \frac{1}{4} (x_2 - x_0)^2$$

$$\frac{1}{4}(x_2-x_0)^2 + \frac{(x_3-x_2)^2}{2} + \sum_{k=3}^{N-1}(x_{k+1}-x_k)^2$$

$$= \frac{1}{4}\left[x_2^2 - 2x_0x_2 + x_0^2 + 2x_3^2 + 2x_2^2 - 4x_2x_3 \right] + \dots$$

$$= \frac{1}{4}\left[3x_2^2 - 3 \cdot 2x_2\left(\frac{x_0+2x_3}{3}\right) + x_0^2 + 2x_3^2 \right] + \dots$$

$$= \frac{1}{4}\left[3\left(x_2 - \frac{x_0+2x_3}{3}\right)^2 \underline{- \frac{(x_0+2x_3)^2}{3} + \frac{3x_0^2+6x_3^2}{3}} \right] + \dots$$

$$= \frac{3x_0^2+6x_3^2-x_0^2-4x_3^2-4x_0x_3}{12} = \frac{1}{6}\left(x_0^2+x_3^2-2x_0x_3\right)$$

$$= \frac{1}{3}\frac{(x_3-x_0)^2}{2} + \sum_{k=3}^{N-1}\frac{(x_{k+1}-x_k)^2}{2}$$

$$= \frac{1}{3}\frac{(x_3-x_0)^2}{2} + \frac{(x_4-x_3)^2}{2} = \frac{1}{2}\left(\frac{1}{3}+1\right)x_3^2 + \dots$$

$$= \frac{1}{2}\cdot\frac{4}{3}\,x_3^2$$

$$\int dx_1\, e^{\frac{im}{\hbar\delta}\frac{1}{2}\frac{2}{1}(x_1-\dots)^2} \int dx_2\, e^{\frac{im}{\hbar\delta}\frac{1}{2}\frac{3}{2}(x_2-\dots)^2} \int dx_3\, e^{\frac{im}{\hbar\delta}\frac{1}{2}\frac{4}{3}(x_3-\dots)^2} \dots \int dx_{N-1}$$

$$\times\; \underline{e^{\frac{im}{\hbar\delta 2N}(x_N-x_0)^2}} = e^{\frac{im}{\hbar T}\frac{(x_N-x_0)^2}{2}}$$

$$\left(\sqrt{\frac{2\pi\hbar\delta}{-im}}\right)^{N-1}\sqrt{\frac{(N-1)!}{N!}} = \left(\sqrt{\frac{2\pi i\hbar\delta}{m}}\right)^{N-1}\sqrt{\frac{1}{N}}$$

$$K = A \sqrt{\frac{2\pi i\hbar\delta}{m}}^{N-1}\sqrt{\frac{1}{N}}\, e^{\frac{im}{\hbar T}\frac{(x_N-x_0)^2}{2}}$$

$$\Rightarrow \quad A = \sqrt{\frac{m}{2\pi i\hbar\delta}}^{N}$$

compare with known expression

Evolution operator

$$\hat{U}(t_f,t_i) = e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}} = \underbrace{e^{-\frac{i}{\hbar}t_f\hat{H}}}_{\hat{U}(t_f,0)}\ \underbrace{e^{\frac{i}{\hbar}t_i\hat{H}}}_{\hat{U}(0,t_i)}$$

$$\hat{U}(t_i,t_f) = \hat{U}^+(t_f,t_i) \quad \text{and} \quad \hat{U}(t_f,t_i)\hat{U}(t_i,t_f) = \mathbb{1}$$

$$\hat{U}\hat{U}^+ = \mathbb{1} \quad \text{unitary}$$

$$i\hbar\partial_{t_f}\hat{U} = \hat{H}_{(t_f)}\hat{U} \quad , \quad |\psi(t)\rangle = \hat{U}|\psi(0)\rangle$$

$$|x_i,t_i\rangle := \hat{U}^+(t_i,0)|x_i\rangle \qquad \hat{X}_H(t) := \hat{U}^+\hat{X}\hat{U}$$

$$\hat{X}_H(t)|x,t\rangle = \hat{U}^+\hat{X}\hat{U}\hat{U}^+|x\rangle = x\hat{U}^+|x\rangle = x|x,t\rangle$$

$$\langle x_f,t_f|x_i,t_i\rangle = \langle x_f|\hat{U}(t_f,0)\hat{U}^+(t_i,0)|x_i\rangle$$

$$= \langle x_f|\hat{U}(t_f,t_i)|x_i\rangle$$