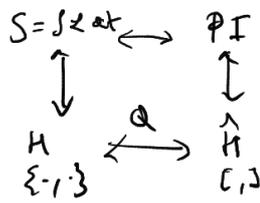


# Lecture 2: Recap



$$\langle x_f | \hat{U}(t_f, t_i) | x_i \rangle$$

$$\langle x_f, t_f | x_i, t_i \rangle = K(x_f, t_f; x_i, t_i) = \int_{\gamma(t_i)=x_i}^{\gamma(t_f)=x_f} \mathcal{D}[\gamma] e^{i \frac{S[\gamma]}{\hbar}}$$

$$\lim_{N \rightarrow \infty} A(N) \int \dots \int dx_1 \dots dx_N \sim \sum_{\gamma}$$

For free particle, we worked out that

$$K(x_N, T, x_0, 0) = \sqrt{\frac{m}{2\pi i \hbar T}} e^{i \frac{m}{2\hbar T} (x_N - x_0)^2}$$

$$S[\gamma] = \int_0^T \frac{1}{2} m \dot{\gamma}(t)^2 dt \quad \text{is the free-particle action}$$

but

$$\gamma_d(t) = x_N \frac{t}{T} + x_0 \left(1 - \frac{t}{T}\right), \quad \dot{\gamma}_d = \frac{x_N - x_0}{T}$$

$$S[\gamma_d] = \int_0^T \frac{1}{2} m \left(\frac{x_N - x_0}{T}\right)^2 dt = \frac{m(x_N - x_0)^2}{2T} \quad \leftarrow \text{classical action}$$

$$K(x_N, T, x_0, 0) = \sqrt{\frac{m}{2\pi i \hbar T}} e^{i \frac{S[\gamma_d]}{\hbar}} \quad K(0, T, 0, 0)$$

and this is exactly the case for separable Lagrangians.  
(e.g. harmonic oscillator)

Matrix elements of operators (analytic in the position operator)

$$\langle x_f, t_f | \hat{X}(t_0) | x_i, t_i \rangle = \int dx_0 \langle x_f, t_f | \hat{X}(t_0) | x_0, t_0 \rangle \langle x_0, t_0 | x_i, t_i \rangle$$

$$= \int dx_0 \langle x_f, t_f | x_0, t_0 \rangle \langle x_0, t_0 | x_i, t_i \rangle x_0$$

$$= \int dx_0 \int_{t_0 \rightarrow t_f} \mathcal{D}[x] e^{\frac{iS[x]}{\hbar}} \int_{t \rightarrow t_0} \mathcal{D}[x] e^{\frac{iS[x]}{\hbar}} x(t_0) = \int \mathcal{D}[x] x(t_0) e^{\frac{iS[x]}{\hbar}}$$

Heaviside function



With two position operators

$$\int \mathcal{D}[x] x(t_2) x(t_1) e^{\frac{iS[x]}{\hbar}} = \langle x_f, t_f | \hat{X}(t_2) \hat{X}(t_1) | x_i, t_i \rangle \theta(t_2 - t_1)$$

$$+ \langle x_f, t_f | \hat{X}(t_1) \hat{X}(t_2) | x_i, t_i \rangle \theta(t_1 - t_2)$$

$$= \langle x_f, t_f | T[\hat{X}(t_1) \hat{X}(t_2)] | x_i, t_i \rangle$$

time-ordered

In general, any analytic function of the position operator

$$O(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\langle x_f, t_f | \hat{O}(x(t_0)) | x_i, t_i \rangle = \int \mathcal{D}[x] O(x(t_0)) e^{\frac{iS[x]}{\hbar}}$$

and

time-ordered

$$\langle x_f, t_f | T[\hat{O}_1(x(t_1)) \dots \hat{O}_n(x(t_n))] | x_i, t_i \rangle$$

$$= \int \mathcal{D}[x] O_1(x(t_1)) \dots O_n(x(t_n)) e^{\frac{iS[x]}{\hbar}}$$

# Ehrenfest theorem

(change of variable)

The path integral is invariant under an overall shift

$$\bar{x}(t) = x(t) + \varepsilon y(t) \quad \text{with} \quad y(t_i) = 0 \quad \text{and} \quad y(t_f) = 0$$

where  $y(t)$  is a "constant" shift (equal for every path  $x(t)$ )

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S[x(t)]} = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x+y] e^{\frac{i}{\hbar} S[x+\varepsilon y]}$$

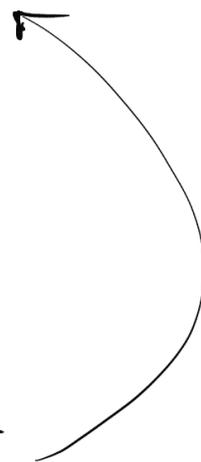
$$= \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} S[x+\varepsilon y]}$$

$$0 = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int \mathcal{D}[x] e^{\frac{i}{\hbar} S[x+\varepsilon y]} = \int \mathcal{D}[x] e^{\frac{i}{\hbar} S[x]} \frac{i}{\hbar} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} S[x+\varepsilon y]$$

$$= \int \mathcal{D}[x] e^{\frac{i}{\hbar} S[x]} \int_{t_i}^{t_f} y \left( \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) dt$$

because

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} S[x+\varepsilon y] &= \int_{t_i}^{t_f} \mathcal{L}(x+\varepsilon y, \dot{x}+\varepsilon \dot{y}) dt \\ &= \int_{t_i}^{t_f} \left( y \frac{\partial \mathcal{L}}{\partial x} + \dot{y} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) dt \\ &= \int_{t_i}^{t_f} y \left( \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) dt \end{aligned}$$



but  $q(t)$  is arbitrary, as  $q(t) = \delta(t - t_1)$   $t_i \leq t_1 \leq t_f$

$$0 = \int \mathcal{D}[x] e^{\frac{i}{\hbar} \delta S[x]} \left[ \frac{\partial \mathcal{L}}{\partial x}(t_1) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}(t_1) \right]$$

$$= \langle x_f, t_f | \hat{\delta S}(t_1) | x_i, t_i \rangle, \quad \forall t_1 \in (t_i, t_f)$$

$$\Rightarrow \langle \psi | \hat{\delta S} | \psi \rangle = 0$$

### Evolution operator

$$\hat{U}(t_f, t_i) = e^{-\frac{i}{\hbar} (t_f - t_i) \hat{H}} = e^{-\frac{i}{\hbar} t_f \hat{H}} e^{\frac{i}{\hbar} t_i \hat{H}}$$

$$\hat{U}(t_f, 0) \hat{U}(0, t_i)$$

$$\hat{U}(t_i, t_f) = \hat{U}^\dagger(t_f, t_i) \quad \text{and} \quad \hat{U}(t_f, t_i) \hat{U}(t_i, t_f) = 1$$

$$\hat{U} \hat{U}^\dagger = 1 \quad \text{unitary}$$

$$i\hbar \frac{\partial}{\partial t} \hat{U} = \hat{H}_{(t)} \hat{U}, \quad |\psi(t)\rangle = \hat{U} |\psi(0)\rangle$$

$$|x_i, t_i\rangle := \hat{U}^\dagger(t_i, 0) |x_i\rangle \quad \hat{X}_H(t) := \hat{U}^\dagger \hat{x} \hat{U}$$

$$\hat{X}_H(t) |x, t\rangle = \hat{U}^\dagger \hat{x} \hat{U} \hat{U}^\dagger |x\rangle = x \hat{U}^\dagger |x\rangle = x |x, t\rangle$$

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | \hat{U}(t_f, 0) \hat{U}^\dagger(t_i, 0) | x_i \rangle$$

$$= \langle x_f | \hat{U}(t_f, t_i) | x_i \rangle$$

Partition function and analogy with statistical mechanics  
(equilibrium thermodynamics)

$$K(x_f, x_i; t_f - t_i) = \langle x_f | \hat{U}(t_f - t_i) | x_i \rangle$$

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x] e^{i \frac{S[x]}{\hbar}} \quad \parallel \quad e^{-\frac{i}{\hbar}(t_f - t_i) \hat{H}}$$

$$Z(t) = \int dx K(x, x, t) = \int dx \langle x | \hat{U}(t) | x \rangle$$

$$= \int_{x(0)=x(t)} \mathcal{D}[x] e^{i \frac{S[x]}{\hbar}} = \text{tr} \hat{U}(t) = \sum_n e^{-\frac{i}{\hbar} t E_n}$$

(periodic paths) or of period  $t$   $x(-t/2) = x(t/2)$

sum over all energy-eigenstates

$$S_t[x] = \int_{-t/2}^{t/2} dz \mathcal{L}(x(z), \dot{x}(z)) dt = \int_{-t/2}^{t/2} dz \left[ \frac{1}{2} \left( \frac{dx}{dz} \right)^2 - V(x) \right]$$

By a formal "rotation of time" (Wick rotation)

$$t = -i \hbar \beta, \quad \beta \in \mathbb{R}$$

we make contact with statistical mechanics  $\frac{1}{k_B T}$

$$Z_E(\beta) = Z(-i \hbar \beta) = \sum_n e^{-\beta E_n}$$

partition function

$$S_{-it\beta} [x] = \int_{+it\beta/2}^{-it\beta/2} dz \left[ \frac{1}{2} \left( \frac{dx}{dz} \right)^2 - V(x) \right]$$

$$\begin{aligned} \sigma = +i\tau, \quad dz = -i d\sigma, \quad \sigma(+it\beta/2) = +it\beta/2 (+i) = -t\beta/2 \\ z = -i\sigma \end{aligned}$$

$$\sigma(-it\beta/2) = -i^2 t\beta/2 = t\beta/2$$

$$S_{-it\beta} [x] = \int_{-t\beta/2}^{t\beta/2} -i d\sigma \left[ \frac{i^2}{2} \left( \frac{dx}{d\sigma} \right)^2 (-i\sigma) - V(x(-i\sigma)) \right]$$

$$= i \int_{-t\beta/2}^{t\beta/2} d\sigma \left[ \left( \frac{dx}{d\sigma} \right)^2_{(-i\sigma)} + V(x(-i\sigma)) \right]$$

$S_E[x]$

$$Z_E(\beta) = \int_{x(-t\beta/2)}^{x(t\beta/2)} \mathcal{D}[x] e^{-\frac{S_E[x]}{\hbar}}$$

$$t \longrightarrow -ipt$$

$$ds^2 = dx^2 - dt^2 = dx^2 + t^2 dp^2$$

↑  
Minkowski

↑  
Euclidean

$S$  (action)  
dynamics  
amplitudes

$S_E$  (classical energy)  
statics (equilibrium)  
probabilities

# Functional methods (generating function)

$$K_{fi}[j] = \langle x_f, t_f | T e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt j(t) \hat{x}_H(t)} | x_i, t_i \rangle$$
$$= N_{fi} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} (S[x] + \langle j, x \rangle)}$$

this is convenient because:

$$\left. \frac{\hbar}{i} \frac{\delta K_{fi}}{\delta j(t_0)} \right|_{j=0} = \langle x_f, t_f | \hat{x}_H(t_0) | x_i, t_i \rangle$$
$$= N_{fi} \int_{x_i}^{x_f} \mathcal{D}[x] x(t_0) e^{\frac{i}{\hbar} S[x]}$$

In general

$$\left. \left( \frac{\hbar}{i} \right)^n \frac{\delta^n K_{fi}}{\delta j(t_1) \dots \delta j(t_n)} \right|_{j=0} = \langle x_f, t_f | T [\hat{x}(t_1) \dots \hat{x}(t_n)] | x_i, t_i \rangle$$

Another useful quantity

$$Z[j] = \langle 0 | T e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} dt j(t) \hat{x}(t)} | 0 \rangle$$

Notice  $Z[0] = \langle 0 | 0 \rangle = 1$   
 $\uparrow$  by convention

$$\left. \frac{\hbar}{i} \frac{\delta Z}{\delta j(t_0)} \right|_{j=0} = \langle 0 | \hat{x}(t_0) | 0 \rangle$$

$$\left( \frac{\hbar}{i} \right)^n \frac{\delta^n Z}{\delta j(t_1) \dots \delta j(t_n)} \Big|_{j=0} = \langle 0 | T \hat{x}(t_1) \hat{x}(t_2) \dots \hat{x}(t_n) | 0 \rangle$$

"correlation function"

What is the path integral representation of this thing?

$$\begin{aligned} \langle x_f, t_f | x_i, -t_f \rangle &= \langle x_f | U(t_f, -t_f) | x_i \rangle \\ &= \langle x_f | e^{-\frac{i}{\hbar} t \hat{H}} | x_i \rangle \end{aligned}$$

$$\left( \text{If } \{ |n\rangle \}_{n \in \mathbb{N}} \text{ is such that } \hat{H} |n\rangle = E_n |n\rangle \right)$$

$$= \sum_{n \in \mathbb{N}} \langle x_f | e^{-\frac{i}{\hbar} t \hat{H}} |n\rangle \langle n | x_i \rangle$$

$$= \sum_{n \in \mathbb{N}} e^{-\frac{itE_n}{\hbar}} \langle x_f | n \rangle \langle n | x_i \rangle$$

$$= \sum_{n \in \mathbb{N}} \psi_n(x_f) \psi_n^*(x_i) e^{-\frac{itE_n}{\hbar}}$$

If  $t \rightarrow \infty$  (and analytically continuing  
from where the above was derived)  
or  $\hbar \rightarrow 0$

$$E_n(\epsilon) = E_n(1 - i\epsilon)$$

$$\simeq \psi_0(x_f) \psi_0^*(x_i) e^{-\frac{it}{\hbar} E_0} \left[ 1 + \mathcal{O}\left( e^{-\frac{it}{\hbar} (E_1 - E_0)} \right) \right]$$

This motivates the formal PI

$$Z[j] = \lim_{t \rightarrow \infty} \int_{x(-\frac{t}{2})=x(\frac{t}{2})} \mathcal{D}[x] e^{\frac{i}{\hbar} S_t[x, j]}$$

$$\frac{\int_{x(-\frac{t}{2})=x(\frac{t}{2})} \mathcal{D}[x] e^{\frac{i}{\hbar} S_t[x]}}$$

The idea now is that

$$\langle 0 | \hat{x}(t_0) | 0 \rangle = \frac{\frac{1}{i} \frac{\delta Z}{\delta j(t_0)} \Big|_{j=0}}{Z} = \frac{\int \mathcal{D}[x] x(t_0) e^{\frac{i}{\hbar} S_0[x]}}{\int \mathcal{D}[x] e^{\frac{i}{\hbar} S_0[x]}}$$

⋮

## Perturbation Theory

Imagine we know exactly how to solve (compute PI) for a quadratic action  $S_0$  (typically the free-particle or harmonic oscillator) and study

$$S = S_0 + \lambda S_1$$

↑  
interaction

Ex:

$$S = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} \dot{x}^2 - \lambda x^4 \right]$$

anharmonic term

Define

$$K_{fi}^0[J] := \int_{x_i}^{x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} \left( S_0 + \int_{t_i}^{t_f} dz j(z) x(z) \right)}$$

$$K_{fi}[J] = \int_{x_i}^{x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} [S_0 + (j, x)]} e^{-\frac{i}{\hbar} \lambda \int_{t_i}^{t_f} x^4 dt}$$

The trick here is to write

$$\int \mathcal{D}[x] F[x] e^{\frac{i}{\hbar} (S_0 + (j, x))} = F \left[ \frac{t \delta}{i \delta j} \right] K_{fi}^0[J]$$

Let  $F[q] = e^{-\frac{i}{\hbar} \lambda \int q^4 dt}$

Then

$$K_{fi}[j] = e^{-\frac{i}{\hbar} \lambda \int \left( \frac{t \delta}{i \delta j(t)} \right)^4 dt} K_{fi}^0[J]$$

$$= K_{fi}^0[j] - \frac{i}{\hbar} \lambda \int \left( \frac{t \delta}{i \delta j} \right)^4 K_{fi}^0[j] dt + O(\lambda^2)$$

It turns out that (see below)

$$K_{fi}^0[j] = K_{fi}^0[0] e^{\frac{i}{\hbar} \left( \frac{1}{2} j * \Delta * j + \langle x_Q, j \rangle \right)}$$

where  $j * \Delta * j = \int_{t_i}^{t_f} \int_{t_i}^{t_f} dz d\sigma j(\sigma) \Delta(z, \sigma) j(z)$

$$\langle x_Q, j \rangle = \int_{t_i}^{t_f} dz x_Q(z) j(z)$$

← classical trajectory

where  $\Delta(z, \sigma)$  is the Green's function  
of the Euler-Lagrange equation of the unperturbed action

$$\partial_z^2 \Delta = \delta(z - \sigma) \quad \text{and} \quad \begin{cases} \Delta(t_i, \sigma) = 0 \\ \Delta(t_f, \sigma) = 0 \end{cases} \quad \forall \sigma$$

Dirichlet BC

Computation of  $K_{fi}^0[j] = \int D[x] e^{\frac{i}{\hbar} S_0[x] + \langle j, x \rangle}$

Conceptually, the action is a quadratic form  $S_0[x] = \frac{1}{2} x^T O x$   
plus a linear forcing

$$S[x] = \frac{1}{2} x^T O x + j^T x$$

The idea is to apply a change of variables  $x = y + x_c$

so that linear terms vanish. This is equivalent to

completing the square:

$$\frac{1}{2} x^T O x + j^T x = \underbrace{\frac{1}{2} y^T O y}_{S_0(y)} + \underbrace{\frac{1}{2} x_c^T O x_c + j^T x_c}_{S(x_c, j)} + \underbrace{j^T y}_{S_{lin}(x_c, y, j)} + \frac{1}{2} (y^T O x_c + x_c^T O y)$$

$$\text{if } x_c = -O^{-1} j \Rightarrow S_{lin}(x_c, y, j) = y^T (O x_c + j) = 0$$

$$\text{and } S(x_c, j) = \frac{1}{2} j^T O^{-1} j - j^T O^{-1} j = -\frac{1}{2} j^T O^{-1} j$$

# Computation of $K_{fi}^0[j]$ for free-particle

$$K_{fi}^0[j] = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} \int_{t_i}^{t_f} \left[ \frac{1}{2} \dot{x}^2 + x(t)j(t) \right] dt}$$

$S[x, j]$

## Change of variables (path parametrization)

$$x(t) = y(t) + x_c(t)$$

↳ fixed path

with  $\begin{cases} y(t_i) = x_i \text{ and } y(t_f) = x_f \\ x_c(t_i) = 0 \text{ and } x_c(t_f) = 0 \end{cases}$

$$S[x, j] = \int_{t_i}^{t_f} \frac{1}{2} \dot{y}^2 dt + \int_{t_i}^{t_f} \left( \frac{1}{2} \dot{x}_c^2 + x_c j \right) dt$$

$S_0[y]$                        $S[x_c, j]$

$$+ \int_{t_i}^{t_f} (\dot{y} \dot{x}_c + y j) dt$$

by parts

$$= S_0[y] + S[x_c, j] + y \dot{x}_c \Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} y (-\ddot{x}_c + j) dt$$

$S_{cl}[y, x_c, j]$

The idea is to find  $x_c$  such that the linear term vanishes for all  $y$  i.e.  $x_c$  is a solution to the inhomogeneous DE:

$$-\ddot{x}_c + \dot{x}_c = 0 \quad \text{and} \quad x_c(t_i) = 0 = x_c(t_f)$$

To highlight the general method, we introduce the Green's function  $\Delta(t, \sigma)$  such that

$$\partial_t^2 \Delta(t, \sigma) = \delta(t - \sigma)$$

$$*) \quad \Delta(t_i, \sigma) = 0 = \Delta(t_f, \sigma), \quad \forall \sigma$$

$$*) \quad \Delta(t, \sigma) \text{ continuous at } t = \sigma$$

$$\text{Then, } x_c(t) = \int_{t_i}^{t_f} \Delta(t, \sigma) j(\sigma) d\sigma := \Delta * j$$

We try a separated solution

$$\Delta(t, \sigma) = \begin{cases} A y_L(t) y_R(\sigma), & t \leq \sigma \\ A y_L(\sigma) y_R(t), & t > \sigma \end{cases}$$

$$\text{where } \left| \begin{array}{l} y_L'' = 0 \quad \text{and} \quad y_L(t_i) = 0 \\ y_R'' = 0 \quad \text{and} \quad y_R(t_f) = 0 \end{array} \right.$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\sigma-\varepsilon}^{\sigma+\varepsilon} \partial_t^2 \Delta \, dt = 1$$

$$\Rightarrow \partial_t \Delta(\sigma+\varepsilon, \sigma) - \partial_t \Delta(\sigma-\varepsilon, \sigma) = 1$$

$$\Rightarrow A \left[ y_R'(\sigma) y_L(\sigma) - y_L'(\sigma) y_R(\sigma) \right] = 1$$

$$\Rightarrow A = \frac{1}{W(\sigma)}$$

where  $W(\sigma) = \begin{vmatrix} y_L(\sigma) & y_R(\sigma) \\ y_L'(\sigma) & y_R'(\sigma) \end{vmatrix}$ , the Wronskian

We check that the Wronskian is constant.

$$\frac{d}{d\sigma} W = \underbrace{y_R'' y_L}_{\rightarrow 0} + \cancel{y_R' y_L'} - \underbrace{y_L'' y_R}_{\rightarrow 0} - \cancel{y_R' y_L'} = 0 \neq$$

$$\text{Let } y_L(t) = C_L(t-t_i)$$

$$y_R(t) = C_R(t_f-t)$$

$$W(\sigma) = -C_R C_L (t_f - t) - C_L C_R (t - t_i)$$

$$= -C_R C_L (t_f - t_i)$$

$$\Delta(t, \sigma) = \begin{cases} -\frac{(t-t_i)(t_f-\sigma)}{t_f-t_i}, & t \leq \sigma \\ -\frac{(t_f-t)(\sigma-t_i)}{t_f-t_i}, & t > \sigma \end{cases}$$

Finally, 
$$x_c(t) = \int_{t_i}^{t_f} \Delta(t, \sigma) j(\sigma) d\sigma$$

With this choice  $S[x_c, j] = 0$ ,  $\forall y$

The remaining terms are

$$\begin{aligned}
 1) \quad S[x_c, j] &= \int_{t_i}^{t_f} \left( \frac{1}{2} \dot{x}_c^2 + x_c j \right) dt \\
 &\stackrel{\text{parts}}{=} \underbrace{x_c \dot{x}_c \Big|_{t_i}^{t_f}}_{=0} + \int_{t_i}^{t_f} x_c \left[ -\frac{1}{2} \ddot{x}_c + j \right] dt \\
 &= \frac{1}{2} \int_{t_i}^{t_f} x_c j dt = \frac{1}{2} \int_{t_i}^{t_f} \int_{t_i}^{t_f} j(t) \Delta(t, \sigma) j(\sigma) d\sigma dt \\
 &:= \frac{1}{2} j * \Delta * j
 \end{aligned}$$

$$\begin{aligned}
 2) \quad y \dot{x}_c \Big|_{t_i}^{t_f} &= y(t_f) \dot{x}_c(t_f) - y(t_i) \dot{x}_c(t_i) \\
 &= x_f \int_{t_i}^{t_f} \partial_t \Delta(t_f, \sigma) j(\sigma) d\sigma - x_i \int_{t_i}^{t_f} \partial_t \Delta(t_i, \sigma) j(\sigma) d\sigma \\
 &= x_f \int_{t_i}^{t_f} \frac{\sigma - t_i}{t_f - t_i} j(\sigma) d\sigma + x_i \int_{t_i}^{t_f} \frac{t_f - \sigma}{t_f - t_i} j(\sigma) d\sigma
 \end{aligned}$$

$$= \int_{t_i}^{t_f} x_d(\sigma) j(\sigma) d\sigma := \langle x_d, j \rangle$$

where  $x_d(\sigma) = x_f \frac{\sigma - t_i}{t_f - t_i} + x_i \frac{t_f - \sigma}{t_f - t_i}$

is the classical trajectory  $\ddot{x}_d = 0$  and  $x_d(t_i) = x_i$   
 $x_d(t_f) = x_f$

So

$$S[x, j] = S_0[y] + \langle x_d, j \rangle + \frac{1}{2} j^* \Delta^* j$$

unforced
boundary term
Quadratic term

hencefore

$$K_{fi}^0[j] = \int D[x] e^{i S[x]}$$

$$= e^{\frac{i}{\hbar} (\langle x_d, j \rangle + \frac{1}{2} j^* \Delta^* j)} \int D[y] e^{\frac{i}{\hbar} S_0[y]}$$

$$K_{fi}^0[j] = e^{\frac{i}{\hbar} (\langle x_d, j \rangle + \frac{1}{2} j^* \Delta^* j)} K_{fi}^0[0]$$

Computation of  $K_{fi}^0[j]$  (for the Harmonic oscillator)

$$K_{fi}^0[j] = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x] e^{\frac{i}{\hbar} \int_{t_i}^{t_f} \left( \frac{1}{2} \dot{x}^2 - \frac{\omega^2}{2} x^2 + xj \right) dt}$$

$S[x]$

Change of variable:

$$x(t) = x_c(t) + y(t)$$

\*  $x_c(t)$  is a "fixed path" with  $x_c(t_i) = 0$  and  $x_c(t_f) = 0$

\*  $y(t)$  is a new "variable path" with  $y(t_i) = x_i$  and  $y(t_f) = x_f$

$S_0[y]$

$$S[x] = \int_{t_i}^{t_f} \left( \frac{1}{2} \dot{y}^2 - \frac{\omega^2}{2} y^2 \right) dt + \int_{t_i}^{t_f} \left( \frac{1}{2} \dot{x}_c^2 - \frac{\omega^2}{2} x_c^2 + x_c j \right) dt$$

$S[x_c, j]$

$$+ \int_{t_i}^{t_f} \left( \dot{y} \dot{x}_c - \omega^2 x_c y + y j \right) dt$$

$$= S_0[y] + S[x_c, j] + y \dot{x}_c \Big|_{t_i}^{t_f}$$

$\left. \vphantom{y \dot{x}_c} \right\}$  Boundary term

$$- \int_{t_i}^{t_f} y \left[ \ddot{x}_c + \omega^2 x_c - j \right] dt$$

$S_{sh}[y, x_c, j]$

would verify if  $x_c(t)$  is the solution to the inhomogeneous ODE:

$$\ddot{x}_c + \omega^2 x_c = \bar{f} \quad \text{with} \quad x_c(t_i) = 0 \quad \text{and} \quad x_c(t_f) = 0$$

for simplicity (and without loss of generality)

let's have  $\tau = t - t_i$  and  $T = t_f - t_i \Rightarrow$

$$x_c(0) = 0$$
$$x_c(T) = 0$$

## Green's function

We look for the Green's function  $\Delta(\tau, \sigma)$

such that

$$\partial_\tau^2 \Delta + \omega^2 \Delta = \delta(\tau - \sigma)$$

so that

$$x_c(\tau) = \int_0^T d\sigma \bar{f}(\sigma) \Delta(\tau, \sigma)$$

and  $\Delta(0, \sigma) = 0 = \Delta(T, \sigma)$ ,  $\forall \sigma$

and  $\Delta(\tau, \sigma)$  is continuous at  $\tau = \sigma$ .

We try a separated solution

$$\Delta(z, \sigma) = \begin{cases} A y_L(z) y_R(\sigma), & z \leq \sigma \\ A y_L(\sigma) y_R(z), & z > \sigma \end{cases}$$

where  $\ddot{y}_L + \omega^2 y_L = 0$  and  $y_L(0) = 0$

$\ddot{y}_R + \omega^2 y_R = 0$  and  $y_R(T) = 0$

$$\lim_{\varepsilon \rightarrow 0} \int_{\sigma-\varepsilon}^{\sigma+\varepsilon} (\partial_z^2 \Delta + \omega^2 \Delta) dz = 1$$

$$\Rightarrow \lim_{\varepsilon \rightarrow 0} \partial_z \Delta(\sigma+\varepsilon, \sigma) - \partial_z \Delta(\sigma-\varepsilon, \sigma) = 1$$

$$\Rightarrow A [y_R'(\sigma) y_L(\sigma) - y_L'(\sigma) y_R(\sigma)] = 1$$

$$\Rightarrow A = \frac{1}{y_R'(\sigma) y_L(\sigma) - y_L'(\sigma) y_R(\sigma)} = \frac{1}{W(y_L, y_R)}$$

$W = \begin{vmatrix} y_L & y_R \\ y_L' & y_R' \end{vmatrix}$  is the so-called Wronskian.

We verify that  $W(\sigma)$  is constant

$$\frac{d}{d\sigma} W(\sigma) = y_R'' y_L + \cancel{y_R' y_L'} - \cancel{y_L' y_R'} - y_L'' y_R$$

$$= \omega^2 y_R y_L - \omega^2 y_L y_R = 0 \quad \neq$$

$$\text{Let } y_L(z) = C_L \sin \omega z \Rightarrow \ddot{y}_L + \omega^2 y_L = 0 \text{ and } y_L(0) = 0$$

$$y_R(z) = -C_R \sin[\omega(T-z)] \Rightarrow \ddot{y}_R + \omega^2 y_R = 0 \text{ and } y_R(T) = 0$$

$$W = \begin{vmatrix} C_L \sin \omega \sigma & -C_R \sin[\omega(T-\sigma)] \\ \omega C_L \cos \omega \sigma & \omega C_R \cos[\omega(T-\sigma)] \end{vmatrix}$$

$$= \omega C_R C_L \left( \sin \omega \sigma \cos[\omega(T-\sigma)] + \cos \omega \sigma \sin \omega(T-\sigma) \right)$$

$$= \omega C_R C_L \sin[\omega \sigma + \omega T - \omega \sigma]$$

$$= \omega C_R C_L \sin \omega T$$

$$\text{So } \Delta(z, \sigma) = \begin{cases} - \frac{\sin \omega z \sin[\omega(T-\sigma)]}{\sin \omega T} & z \leq \sigma \\ - \frac{\sin \omega \sigma \sin[\omega(T-z)]}{\sin \omega T} & z > \sigma \end{cases}$$

$$\text{Finally, } x_c(z) = \int_0^T d\sigma \Delta(z, \sigma) j(\sigma)$$

With this choice,  $S_{lin}[y, x_c, j] = 0$ ,  $\forall y$

The remaining terms are:

$$\begin{aligned}
 1) \quad S[x_c, j] &= \int_0^T \left( \frac{1}{2} \dot{x}_c^2 - \frac{\omega^2}{2} x_c^2 + x_c j \right) dz \\
 &= \underbrace{\frac{1}{2} x_c \dot{x}_c \Big|_0^T}_{\rightarrow 0} - \int_0^T x_c \left[ \underbrace{\frac{1}{2} \ddot{x}_c + \frac{\omega^2}{2} x_c}_{\frac{1}{2} j} - j \right] dz \\
 &= + \frac{1}{2} \int_0^T x_c j dz = \frac{1}{2} \int_0^T \int_0^T j(\tau) \Delta(\tau, \sigma) j(\sigma) \\
 &:= \frac{1}{2} j * \Delta * j
 \end{aligned}$$

$$\begin{aligned}
 2) \quad \gamma \dot{x}_c \Big|_0^T &= \gamma(T) \dot{x}_c(T) - \gamma(0) \dot{x}_c(0) \\
 &= x_f \int_0^T d\sigma \partial_z \Delta(T, \sigma) j(\sigma) - x_i \int_0^T d\sigma \partial_z \Delta(0, \sigma) j(\sigma) \\
 &= x_f \int_0^T d\sigma \frac{\sin \omega \sigma}{\sin \omega T} j(\sigma) + x_i \int_0^T d\sigma \frac{\sin[\omega(T-\sigma)]}{\sin \omega T} j(\sigma) \\
 &= \int_0^T d\sigma \frac{j(\sigma)}{\sin \omega T} \left( x_f \sin \omega \sigma + x_i \sin[\omega(T-\sigma)] \right) \\
 &= \int_0^T d\sigma j(\sigma) x_d(\sigma) \\
 &= \langle x_d, j \rangle
 \end{aligned}$$

In fact, the classical solution is  $x_d(t)$

such that  $\ddot{x}_d + \omega^2 x_d = 0$  and

$x_d(0) = x_i$  and  $x_d(T) = x_f$

Then

$$x_d = x_i \cos \omega t + \frac{(x_f - x_i \cos \omega T)}{\sin \omega T} \sin \omega t$$

$$= \frac{x_i (\cos \omega t \sin \omega T - \cos \omega T \sin \omega t)}{\sin \omega T} + x_f \frac{\sin \omega t}{\sin \omega T}$$

$$x_d(t) = x_i \frac{\sin[\omega(T-t)]}{\sin \omega T} + x_f \frac{\sin \omega t}{\sin \omega T}$$

The reason is Lagrange identity:

$$y \dot{x}_c \Big|_{t_i}^{t_f} = x_f \dot{x}_c(t_f) - x_i \dot{x}_c(t_i) = x_d \dot{x}_d - \dot{x}_d x_c \Big|_{t_i}^{t_f}$$

Lagrange  $\int_{t_i}^{t_f} dt \left[ x_d \underbrace{\left( \frac{d^2}{dt^2} + \omega^2 \right) x_c - x_c \underbrace{\left( \frac{d^2}{dt^2} + \omega^2 \right) x_d}_{=0} \right]$

where  $x_d$  is such that  $\ddot{x}_d + \omega^2 x_d = 0$  and  $x_d(t_i) = x_i, x_d(t_f) = x_f$

the classical solution.

So

$$S[x] = S_0[y] + \langle x \alpha, j \rangle + \frac{1}{2} j^* \Delta^* j$$

unforced
boundary term
Quadratic term

hence

$$K_{fi}^0[j] = \int \mathcal{D}[x] e^{i S[x]}$$

$$= e^{\frac{i}{\hbar} (\langle x \alpha, j \rangle + \frac{1}{2} j^* \Delta^* j)} \int \mathcal{D}[y] e^{\frac{i}{\hbar} S_0[y]}$$

$$K_{fi}^0[j] = e^{\frac{i}{\hbar} (\langle x \alpha, j \rangle + \frac{1}{2} j^* \Delta^* j)} K_{fi}^0[0]$$

Feynman diagrams (pictorial representation of Green's function, to keep track of terms)

Rules:

$$\text{---} \equiv j * \Delta * j = \iint d\tau j(\tau) \Delta(\tau - \sigma) j(\sigma)$$

$$\begin{array}{c} z_1 \\ \bullet \text{---} \end{array} \equiv (j * \Delta)(z_1) = \int d\tau j(\tau) \Delta(\tau - z_1)$$

$$\begin{array}{c} z_1 \quad z_2 \\ \bullet \text{---} \bullet \end{array} \equiv \Delta(z_1 - z_2)$$

$S_0$ ,

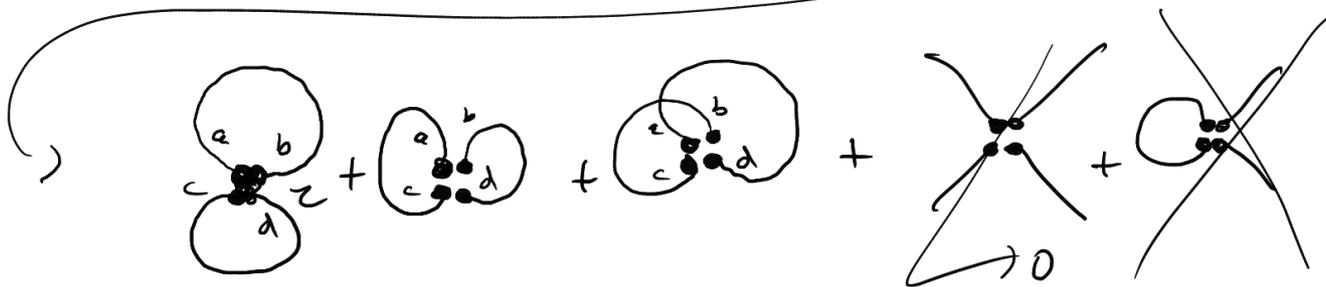
$$\frac{\hbar \delta}{i \delta j(z_1)} e^{\frac{i}{\hbar} j * \Delta * j} \Big|_{j=0} = \lim_{j \rightarrow 0} \begin{array}{c} z_1 \\ \bullet \text{---} \end{array} e^{\frac{i}{\hbar} j * \Delta * j} = 0$$

$$\frac{\delta^2}{\delta j(z_1) \delta j(z_2)} e^{\frac{i}{\hbar} j * \Delta * j} \Big|_{j=0} = \lim_{j \rightarrow 0} \begin{array}{c} z_1 \quad z_2 \\ \bullet \text{---} \bullet \end{array} + \begin{array}{c} z_1 \\ \bullet \text{---} \\ z_2 \\ \bullet \text{---} \end{array} \rightarrow \Delta(z_1 - z_2)$$

only pairs survive (Issaev's / Wick theorem)

(open diagrams disappear in the limit  $j \rightarrow 0$ )

$$K_{NL} [j \rightarrow 0] = K_{free} \left( 1 - \frac{i\lambda}{\hbar} \int \left( \frac{\hbar \delta}{i} \delta_j(z) \right)^4 e^{\frac{i\lambda}{\hbar} j * \Delta * j} + O(\lambda^2) \right)_{j=0}$$



$$= K_{free} \left( 1 - \frac{i\lambda}{\hbar} 3 \int_{t_i}^{t_f} \Delta^2(z, z) dz + O(\lambda^2) \right)$$

then

$$\langle \hat{x}^2(z_1) \hat{x}(z_3) \hat{x}(z_4) \rangle =$$

$$+ z_1 \circlearrowleft z_3 z_4$$

$$\propto 2 \Delta(z_1 - z_3) \Delta(z_1 - z_4) + \Delta(z_1, z_1) \Delta(z_3 - z_4)$$

$$\langle x^3(z_1) x(z_4) \rangle =$$

$$\propto 3 \Delta^2(z_1 - z_1) \Delta(z_1 - z_4)$$