Scattering events

single scalar field theory

$$\mathcal{L}_{classical} = \underbrace{\mathcal{L}_{Free}}_{} + \underbrace{\mathcal{L}_{Interaction}}_{}$$

$$= \frac{1}{2}(\partial\phi)^2 - \frac{m}{2}\phi^2 - \lambda\phi^4$$

$$\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\rangle = \frac{\int D[\phi]\, \phi(x_1)\phi(x_2)\phi(x_3)\, e^{\frac{i}{\hbar}S[\phi]}}{\int D[\phi]\, e^{\frac{i}{\hbar}S[\phi]}}$$

$|||$

expectation value of operators (represents probability of scattering $\phi + \phi \longleftrightarrow \phi$)

Recipe:

$$\mathcal{L}_J := \mathcal{L}_{Free} + J_{(x)}\phi_{(x)}$$

$$Z(J) := \int D[\phi]\, e^{\frac{i}{\hbar}S_J[\phi]}$$

it turns out

$$= Z(0)\, e^{\frac{1}{2}J*G*J},$$

where $G(x,y)$ is the classical Green's function of the classical free field problem.

and

$$\frac{1}{2}J*G*J = \frac{1}{2}\iint J(x)\,G(x,y)\,J(y)\,dx\,dy$$

Using functional derivative with respect to $J(x_1)$

$$\frac{\delta J(y)}{\delta J(x_1)} \equiv \delta(x_1 - y)$$

we have

$$\frac{\delta Z}{\delta J(x_1)} = \int D[\phi]\, \phi(x_1)\, e^{\frac{i}{\hbar} S_J}$$
$$= [G * J](x_1)\, Z(J)$$

and

$$\frac{\delta^2 Z}{\delta J(x_1)\, \delta J(x_2)} = \int D[\phi]\, \phi(x_1)\, \phi(x_2)\, e^{\frac{i}{\hbar} S_J}$$
$$= G(x_1, x_2)\, Z(J) + [G * J](x_1)\, [G * J](x_2)\, Z(J)$$

$$\frac{\delta^3 Z}{\delta J(x_1)\, \delta J(x_2)\, \delta J(x_3)} = \int D[\phi]\, \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, e^{\frac{i}{\hbar} S_J}$$
$$= G(x_1, x_2)\, [J * G](x_3)\, Z(J)$$
$$+ G(x_1, x_3)\, [J * G](x_2)\, Z(J)$$
$$+ G(x_2, x_3)\, [J * G](x_1)\, Z(J)$$
$$+ [J * G](x_1)\, [J * G](x_2)\, [J * G](x_3)\, Z(J)$$

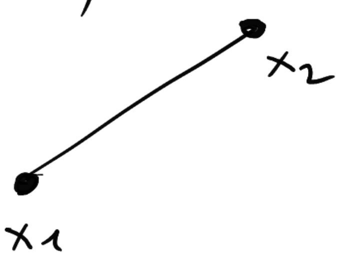
$$\frac{\delta^4 Z}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)\,\delta J(x_4)} = \int \mathcal{D}[\phi]\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\,e^{\frac{i}{\hbar}S_J}$$

$$\begin{aligned}
= Z(J)\Big\{ & G(x_1,x_2)\,G(x_3,x_4) + G(x_1,x_3)\,G(x_2,x_4) + G(x_2,x_3)\,G(x_1,x_4) \\
& + G(x_1,x_2)\,G*J(x_3)\,G*J(x_4) + G(x_1,x_3)\,G*J(x_2)\,G*J(x_4) \\
& + G(x_2,x_3)\,G*J(x_1)\,G*J(x_4) \\
& + G(x_1,x_4)\,G*J(x_2)\,G*J(x_3) + G(x_2,x_4)\,G*J(x_1)\,G*J(x_3) \\
& + G(x_3,x_4)\,G*J(x_1)\,G*J(x_2) \\
& + G*J(x_1)\,G*J(x_2)\,G*J(x_3)\,G*J(x_4) \Big\}
\end{aligned}$$

and so on and so forth for higher order derivatives...

We will need a strategy to handle this algebra more efficiently. This is where Feynman diagrams come into play.

A vertex is a line representing the propagator $G(x_1, x_2)$



We originally wanted to compute

$$\int D[\phi]\ \phi(x_1)\phi(x_2)\phi(x_3)\, e^{\frac{i}{\hbar}S_{free}}\, e^{\frac{i}{\hbar}\lambda\int\phi^4 dx}$$

$\lambda \ll 1$

$$\simeq \int D[\phi]\ \phi(x_1)\phi(x_2)\phi(x_3)\, e^{\frac{i}{\hbar}S_{free}}\left[1+\frac{i\lambda}{\hbar}\int\phi^4 dx + \left(\frac{i\lambda}{\hbar}\right)^2\int\phi^4 dx\int\phi^4 dy + O(\lambda^3)\right]$$

$$= \underline{\int D[\phi]\,\phi(x_1)\phi(x_2)\phi(x_3)\, e^{\frac{i}{\hbar}S_{free}}}$$

$$+ \frac{i\lambda}{\hbar}\underline{\int D[\phi]\ \phi(x_1)\phi(x_2)\phi(x_3)\int\phi^4(x)dx\ e^{\frac{i}{\hbar}S_{free}}}$$

$$+ O(\lambda^2)$$

$$= \underline{\left.\frac{\delta^3 Z}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)}\right|_{J=0}} + \underline{\frac{i\lambda}{\hbar}\left.\int\frac{\delta^7 Z}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J^4(x)}dx\right|_{J=0}} + O(\lambda^2)$$

We observe that this will always involve an odd number of functional derivatives in J. Since $Z(J)$ is even in J, when $J\to 0$, all those terms will vanish. So

$$\langle\hat{\phi}\hat{\phi}\hat{\phi}\rangle = 0$$

No scattering events involving odd number of particles.

Let's instead look at

$$\langle \hat\phi(x_1)\,\hat\phi(x_2)\,\hat\phi(x_3)\,\hat\phi(x_4)\rangle$$

exercise

$$= \left.\frac{\delta^4 Z}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} + \frac{i\lambda}{\hbar}\int \frac{\delta^8 Z}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)\delta^4 J(x)}\,dx + O(\lambda^2)$$

$$= G(x_1,x_2)G(x_3,x_4) + G(x_1,x_3)G(x_2,x_4) + G(x_1,x_4)G(x_2,x_3)$$

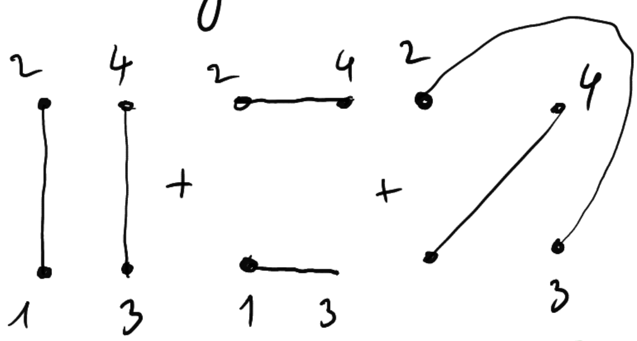
$$+ \frac{i\lambda}{\hbar}\Big\{ \sum_\sigma G_{\sigma_1\sigma_2}G_{\sigma_3\sigma_4}\int G(x,x)G(x,x)\,dx$$

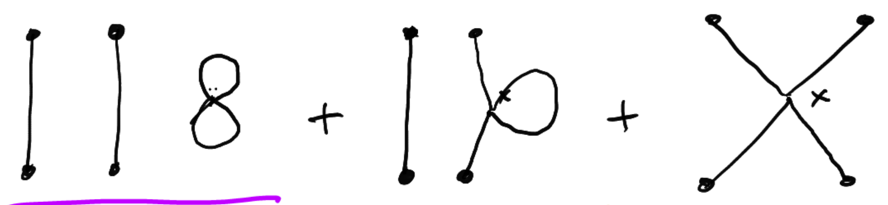
$$+ \sum_\sigma G_{\sigma_1\sigma_2}\int G(x,x_{\sigma_3})G(x,x_{\sigma_4})G(x,x)\,dx$$

$$+ \sum_\sigma \int G(x,x_{\sigma_1})G(x,\sigma_2)G(x,\sigma_3)G(x,\sigma_4)\,dx \Big\}$$

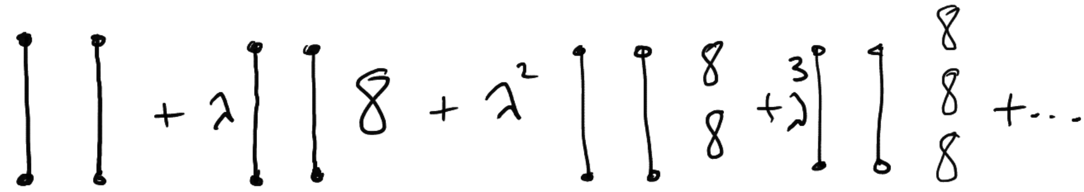
where σ is a permutation of $\{1,2,3,4\}$ and $G_{12} = G(x_1,x_2)$

Pictorially,



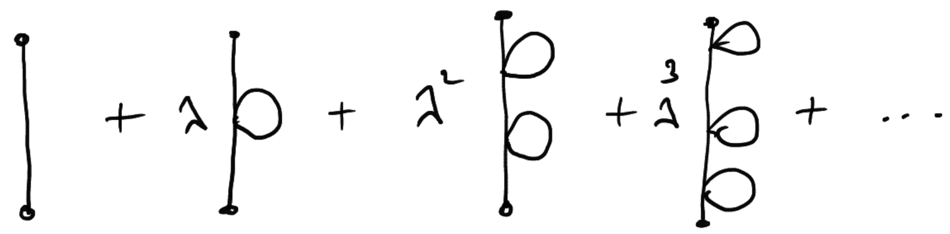
$+\dfrac{i\lambda}{\hbar}\Big\{$  $\Big\}$

Now,



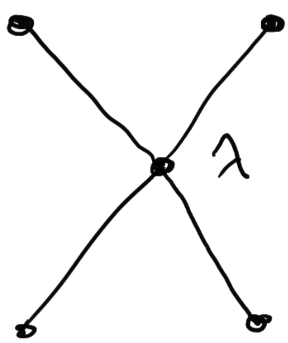
are describing the same process.

Then,



are describing the same process.

So the only interesting "new" diagram at order λ is the 4-point interaction term



It is the fundamental interaction term and is interpreted as the binary collision: $\phi + \phi \longleftrightarrow \phi + \phi$

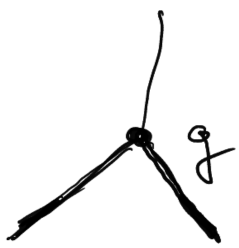
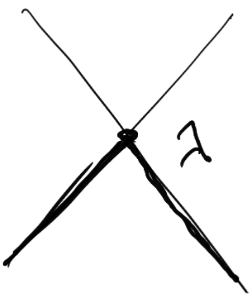
(or $\phi \leftrightarrow \phi + \phi + \phi$ or $\cdot \longleftrightarrow \phi + \phi + \phi + \phi$)

This was for only one interaction term $\frac{\lambda}{4!}\phi^4$

Imagine we have more than one field ϕ_1, ϕ_2 and more than one interaction term... this is going to be tedious! We need a safe way to keep track of diagrams, structure/categorize.

$\Longrightarrow$ <u>Symmetry</u> $\Longleftrightarrow$ constraint on Feynman diagram
$\Longleftrightarrow$ physical/observed events

Motivation: if $S_I[\phi] = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$

 g and  λ

$\phi + \phi \longrightarrow \phi \quad +g$ $\qquad\qquad$ $\phi + \phi \rightarrow \phi + \phi \quad +\lambda$

if $\phi' = -\phi$, renormalisation of the fields by -1.

$\phi' + \phi' \longleftrightarrow \phi' \quad -g$? it would be very weird that the scattering amplitude depended on the sign of ϕ (normalisation)
$\Longrightarrow$ violate time-reversal.

Imposing that $S[\phi] = S[-\phi]$ (mirror)

$\Longrightarrow$ $g=0$ and number of particles conserved in scattering events involving $\phi+\phi$

More complicated example

φ_1, φ_2 two scalar fields

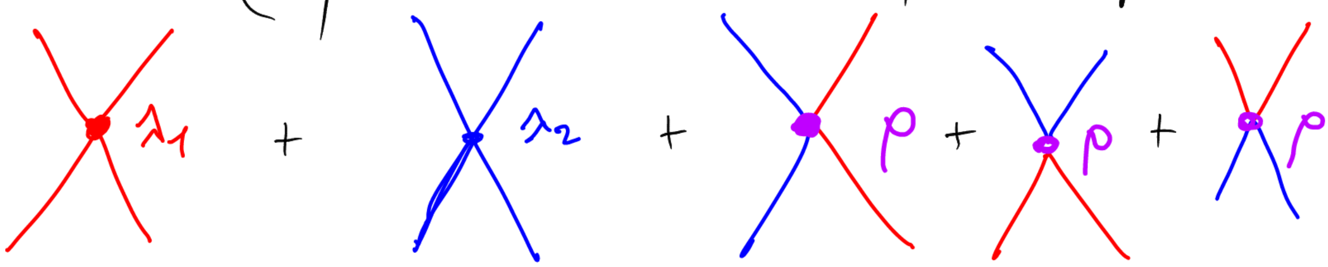
Assume $\mathcal{L}(\varphi_1,\varphi_2) = \mathcal{L}(\varphi_1,-\varphi_2) = \mathcal{L}(-\varphi_1,\varphi_2) = \mathcal{L}(-\varphi_1,-\varphi_2)$

The most general Lagrangian will be of the form (up to φ^4)

$$\mathcal{L} = \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}m_1^2\varphi_1^2 + \frac{1}{2}(\partial\varphi_2)^2 - \frac{m_2^2}{2}\varphi_2^2$$
$$- \frac{\lambda_1}{4}\varphi_1^4 - \frac{\lambda_2}{4}\varphi_2^4 - \frac{\rho}{2}\varphi_1^2\varphi_2^2$$

$$= \frac{1}{2}\,\partial\vec{\varphi}^T\partial\vec{\varphi} - \frac{1}{2}\varphi^T\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}\varphi$$
$$- \frac{\lambda_1}{4}\underline{\varphi_1^4} - \frac{\lambda_2}{4}\underline{\varphi_2^4} - \frac{\rho}{2}\underline{\varphi_1^2\varphi_2^2}$$

$\vec{\varphi} = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$, $m_1, m_2, \lambda_1, \lambda_2, \rho \Rightarrow 5$ parameters



Things would be so harmonious if $m_1 = m_2 = m$
and $\lambda_1 = \lambda_2 = \rho = \lambda$

$$\mathcal{L} = \frac{1}{2}\,\partial\vec{\varphi}^T\partial\vec{\varphi} - \frac{1}{2}m^2\,\vec{\varphi}^T\vec{\varphi} - \frac{\lambda}{4}\left(\vec{\varphi}^T\vec{\varphi}\right)^2$$

ie if the Lagrangian was invariant
under $\vec{\varphi}\,' = R(\theta)\,\vec{\varphi}$ where $R^T R = \mathbb{1}$

Elementary(?) particles – Standard model

leptons

$$e^-,\ e^+$$
$$\nu_e^0,\ \bar{\nu}_e^0$$

hadrons

baryons

p^+ $\bar{p}^-$

n^0 $\bar{n}^0$ 3

mesons

$\pi^-, \pi^0, \pi^+, K^-, K^0, K^+, \bar{K}^0$ 2

colourless

quarks

up	red
down	blue
	green

Fermions

B: 1 1 0 0

L: 0 0 1 -1

$$n^0 \longrightarrow p^+ + e^- + \bar{\nu}_e^0$$

$\Downarrow$

pattern emerge from scattering experiments

$\Downarrow$

constraints on the terms allowed in the action $\Longrightarrow$ symmetry/conservation of "quantum numbers"

Poincaré (space-time) : $\mathcal{L}(\vec{\phi}, \partial\vec{\phi})$ invariant with respect to boosts, rotations

Flavour (internal) : $\underbrace{U(1) \times SU(2)}_{\text{electro-weak}} \times \underbrace{SU(3)}_{\text{strong force}} \cong$ SU(5), E_8, Susy, String

$$\mathcal{L}_{\text{matter}} = \bar{\Psi}\gamma^\mu\left(i\partial_\mu - \frac{g'}{2}Y_W B_\mu - \frac{g}{2}\sigma_i W^i_\mu - g_s T_a G^a_\mu\right)\Psi$$

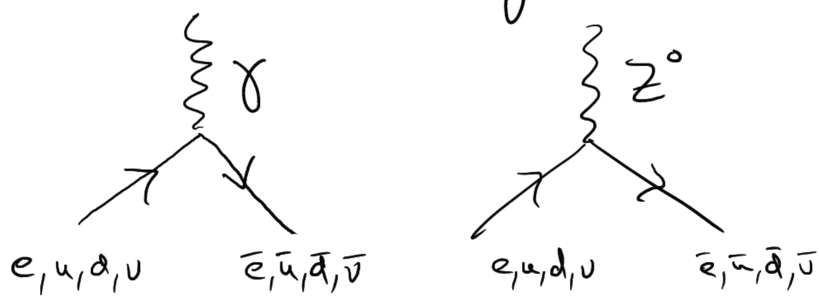
$$\Psi = \begin{pmatrix} e \\ \bar{e} \\ \nu \\ \bar{\nu} \\ u_r \\ \bar{u}_r \\ u_b \\ \bar{u}_b \\ \vdots \\ d_r \\ \bar{d}_r \\ \vdots \end{pmatrix}$$

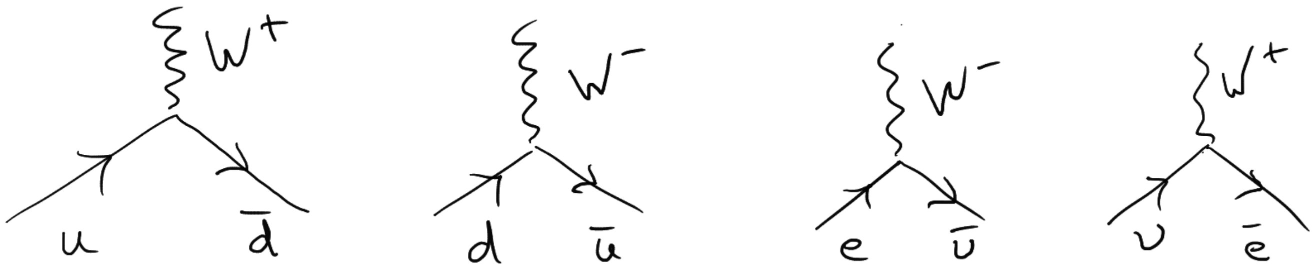
1 number (phase) to "rotate" complex numbers

3 generators to "rotate" 2 $\mathbb{C}$

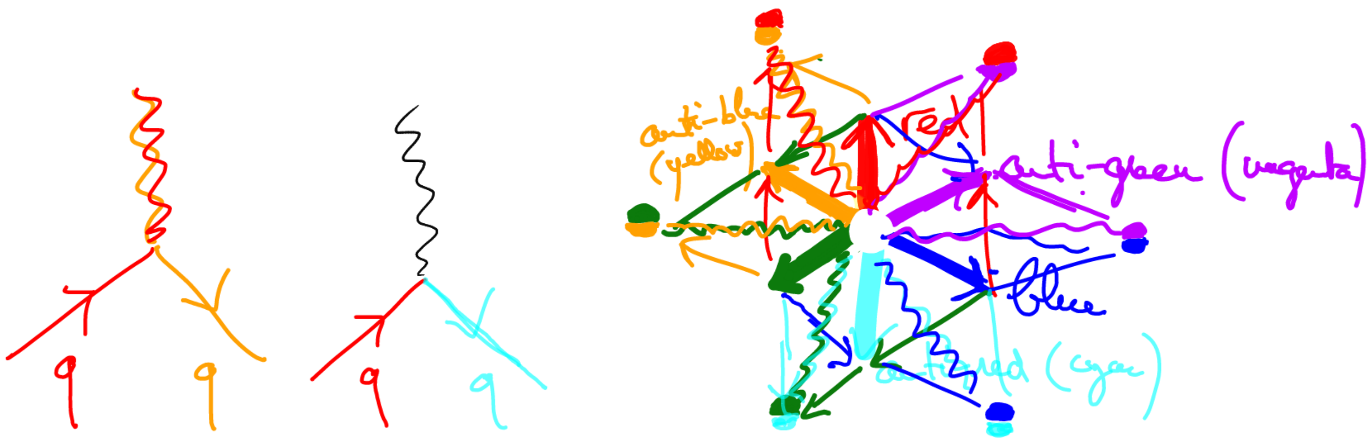
8 generators to "rotate" 3 $\mathbb{C}$

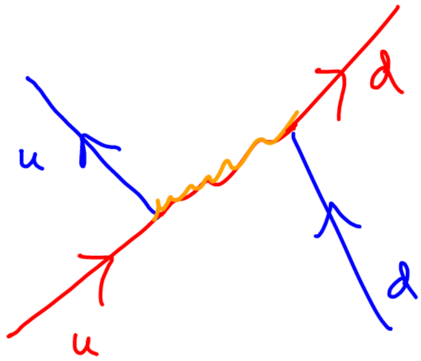
1+3 are γ, W^+, W^-, Z bosons





8 are the gluons such that

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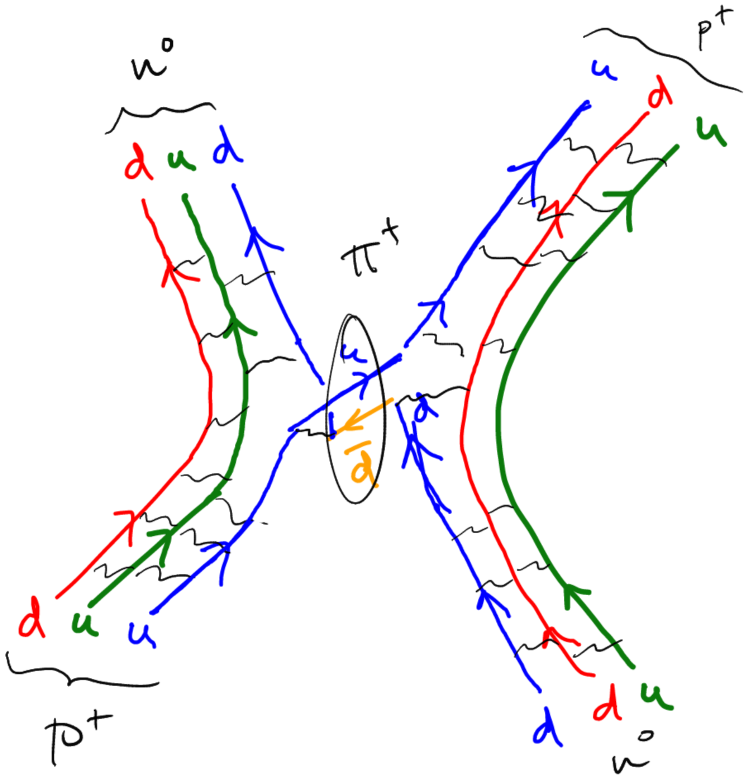
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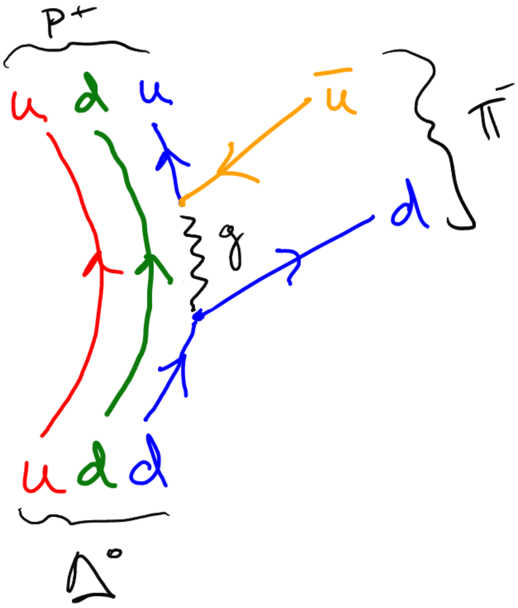
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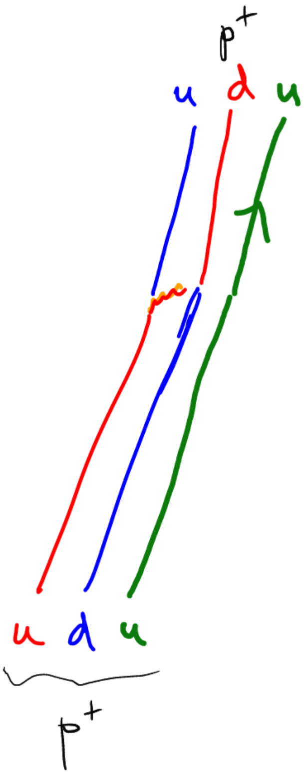
p+
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